Decoding under Integer Metrics Constraints

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Abstract— **We investigate the problem of decoding digital data when soft decisions are constrained to take on values from a finite set. We propose a physically reasonable objective function for selecting the desired assignment of metrics to the received analog signals. We develop a search algorithm for designing a table-look-up that is used by the decoder to select the appropriate intermediate metrics and show that an optimum solution exists. We provide a number of illuminating examples to elucidate our ideas and work out in detail some practical cases.**

I. Introduction

In this paper we examine the problem of decoding coded digital data when soft decisions are constrained to take on values from a finite set of possible metrics (integer values). This is an important practical consideration since actual decoders must inherently use finite precision arithmetic while optimum decision rules require the evaluation of likelihood functions, which for real transmission channels assume a continuum of values.

In order to circumvent these nonphysical specifications, ad-hoc approaches are usually proposed such as, for example, quantizing the received signals or quantizing the continuous decision functions themselves. While these solutions often appear reasonable, the concomitant consequences in terms of performance degradation are extremely difficult to assess. More importantly, the trade-offs between finite computational resources and performance penalties are generally ill understood and a design theory based on finite precision would appear to be very desirable and useful.

Here we address some aspects of this general problem and restrict attention to the design of decoders of digital data transmitted over analog channels. We insist right from the start that decision functions associated with the decoding process must assume only integer[1] values from a given finite set. How to assign (select) these integer-valued decision functions, or metrics, based on reasonable algorithms and how to assess the consequences of the various choices are the central themes of this paper.

Specifically, our approach is the following. Given the requirement that soft decisions in decoding block or convolutionally coded signals must be assigned values from a finite set of integers, we propose and develop algorithms for making such assignments which are based on maximizing a mathematically tractable and physically reasonable cost function. As we shall elucidate in the sequel, the proposed cost function is referred to in the literature as the generalized cut-off rate of the channel [1]. While this channel rate may not be the largest achievable rate, it possesses the important property of guaranteeing that as long as the actual data rate is less than this rate, the probability of error can be made as small as desired by increasing the code length.

Wozencraft and Kennedy [2] were first to suggest that a reasonable modulation system design criterion is the "cut-off rate", R_0, of the Discrete Memoryless Channel (DMC). Since, R_0, is the upper limit of code rates for which the average decoding computation per digit is finite when sequential decoding is used. Massey [3] argued that the appropriate modulation criterion is R_0, and introduced an iterative procedure for finding the optimum quantization boundaries that maximize R_0, for the binary case. His result was generalized by Lee [4] for L-ary modulation. While this approach induces a partition of the likelihood space the decoder still has to assign optimal metrics with infinite precision for the quantization regions. However, in most practical coded systems the decoder must use a finite set of metrics, which are fixed for all channel conditions. In this case the designer's task is to find a "good" partition and metric's assignment of the likelihood space that match the set of metrics. Biederman, Omura and Jain [1] considered the error performance of a coded system where channel statistics can only be approximated. Because of this mismatch, they were led to introduce the Generalized R_0 criteria. In particular, when the decoder is using an integer set of metrics their results reduced to the form of polynomial equations for which Viterbi and Omura [5, pp. 291–292] derived an upper bound on the error performance of a coded system employing a convolutional code.

Since integer metrics are generally not optimum for a given channel, the particular problem addressed here falls into the general category in information theory that deals with the ultimate possible performance of channels with mismatched metrics [6]. The basic question addressed by information theorist is the existence of the largest possible achievable data rate for a mismatched channel. It is still a conjecture that for the memoryless channel with a mismatched decision metric, Hui's capacity [6] cannot be exceeded. However, our problem which focuses mainly on issues dealing with the synthesis of decision rules, has received scant attention in the literature.

While it might be desirable to select integer metrics that maximize Hui's capacity, this is unfortunately an in-

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[1] The derivation in the paper holds for any finite and bounded set of metrics.

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tractable task and, therefore, we have settled on a simpler and mathematically tractable objective function as will be seen in the sequel. When the set of integer metrics is allowed to become unbounded, the objective function approaches the computational cut-off rate of the channel [1]. In this case it is possible to approximate the optimum likelihood function metrics, and it is of course well known that with the optimum metrics, it is possible to achieve Shannon's channel capacity which is always greater than the computational cut-off rate. Since our objective function is always less than capacity, we attribute the short fall to our particular bounding technique as will be seen later.

In Section II, we introduce the channel model and formulate the problem. In Section III, we develop the design algorithm and prove the existence of the optimum metric. Section IV analyses some special illuminating cases and has some numerical examples. Section V reexplores the relation between our metric assignment and the error performance of a particular coded system, Section VI contains our conclusions.

II. Channel Model Motivation and Problem Formulation

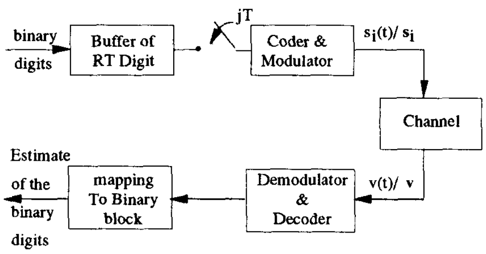

Fig. 1. A general block coded digital data communications system.

For simplicity of presentation, we treat block coded systems but the general ideas apply to convolutionally coded systems as well. A general block coded digital data communications system is depicted in Figure 1. A buffer accepts blocks of $B = RT$ binary digits where R is the input rate in bits/s, and T is the block duration in seconds. For each block, the coder-modulator combination selects one of $M = 2^{RT}$ suitable signals, $s_1(t)$, $i = 1, \ldots, M$, $0 \leq t \leq T$, for transmission over the channel. We presume that it is possible to represent these time functions by vectors, $\mathbf{s}_i$, $i = 1, \ldots, M$, in N'-dimensional space. Usually, N' depends on the time-bandwidth product, or on other suitable signal parameters. The association of the binary data blocks with the signal vectors, $\mathbf{s}_i$, $i, \ldots, M$, is the function of the encoder.

The channel corrupts the input signal, yielding an output waveform which will also be represented by a vector $v(t)$, which will also be represented by a vector $\mathbf{v}$ in N-dimensional space where $N \geq N'$. We presume that we are given the vector-valued probability density function of the channel output vector conditioned on each of the M input vectors, i.e., $p(\mathbf{v}|\mathbf{s}_i)$, $i = 1, \ldots, M$.

At the receiver, the demodulator-decoder combination partitions the entire N-dimensional space into M nonintersecting sets, A_i, $i, \ldots, M$, corresponding to the M input signals. If the received vector $\mathbf{v}$ falls in A_i the decoder, which knows the encoding rule of the coder, provides the output buffer with the binary block corresponding to the signal $\mathbf{s}_i$. If the output binary digits do not correspond exactly to the digits in the input buffer, a "block error" is committed.

As is well known, the optimum selection criterion for the partition set, A_i, $i = 1, \ldots, M$, which maximizes the probability of correct decoding, P_c, is given by the maximum-likelihood-detection rule:

$$\mathbf{v} \in A_i \text{ if } p(\mathbf{v}|\mathbf{s}_i) = \max_j p(\mathbf{v}|\mathbf{s}_j) \quad , \tag{1}$$

where it is assumed that the M possible signals are equally probable. Clearly, this is the optimum metric and its computation requires infinite precision to evaluate. To motivate the use of our objective function, we consider the resulting probability of error when the optimum decision rule is employed.

Letting $G_i(\mathbf{v}) = 1$ when $\mathbf{v} \in A_i$, and 0 otherwise, the probability of error is,

$$P_e = 1 - P_c = \frac{1}{M} \sum_{j=1}^{M} \sum_{\substack{i=1 \\ i \neq j}}^{M} \int p(\mathbf{v}|\mathbf{s}_j)\, G_i(\mathbf{v})\, dv \, . \tag{2}$$

Of course for any code consisting of a set of M vectors $\{\mathbf{s}_i\}$ it is very difficult to evaluate the probability of error exactly and therefore, it is necessary to upper bound (2) and then to assume that the codes are selected at random.

Getting back to the problem at hand, suppose that for some reason instead of decoding with the optimum decision function, or metric, $\log\left(p(\mathbf{v}|\mathbf{s}_i)\right)$ we must decode with a mismatched metric, $\tilde{m}(\mathbf{v}|\mathbf{s}_i)$. The reasons for using a mismatched metric may be varied but here, we require right from the start, that the metrics assume values from a finite set of metrics. This of course is a serious constraint and an increase in the probability or error is expected.

There are many ways to assign metrics to the received signal. We clearly would like to choose such assignments in such a way as to minimize the probability of error and yet not violate the constraints. So, our goal then is to determine a metric assignment for $\mathbf{v}$ from a given set of metrics, which minimize a tight upper bound on the probability of error, where these metrics are used to decide on signal $\mathbf{s}_i$ when

$$\tilde{m}(\mathbf{v}|\mathbf{s}_i) \geq \tilde{m}(\mathbf{v}|\mathbf{s}_j), \quad \text{for all } i \neq j \quad , \tag{3}$$

while $\tilde{m}(\mathbf{v}|\mathbf{s}_i)$ can only assume values from the set of integers $1, 2, \ldots, Q'$, (or any other finite set of metrics) which will be called a metric set. When equality results, a fair coin is tossed to resolve the tie.

To proceed toward the objective function, the constrained integer metrics $\tilde{m}(\mathbf{v}|\mathbf{s}_i)$ are used in a standard

Chernoff bound applied to (2) .

$$P_e(\lambda) \;\le\; \tfrac{1}{M} \sum_{j=1}^{M} \sum_{\substack{i=1 \\ i\neq j}}^{M} \int p(\mathbf{v}|\mathbf{s}_j) \, e^{\lambda\left[\tilde{m}(v|s_i)-\tilde{m}(v|s_j)\right]} \, d\mathbf{v}\,, \lambda > 0\,. \tag{4}$$

This holds since for all $\mathbf{v}$ and $\lambda \ge 0$,

$$G_i(\mathbf{v}) = e^{\lambda\left[\tilde{m}(\mathbf{v}|\mathbf{s}_i)-\tilde{m}(\mathbf{v}|\mathbf{s}_j)\right]}\,, \tag{5}$$

and as is standard in this approach, the positive constant λ may be optimized to yield the tightest bound. To make further progress, we apply a standard random coding argument and average (5) with respect to all sets of code vectors, $\mathbf{s}_i$. It then follows that there must be at least one signal set that achieves a probability of error not greater than the average.

Toward this end, let $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_M$ be selected independently from a population characterized by an identical probability density function, $q(\mathbf{s}_i)$, and $q(\mathbf{s}_i, \mathbf{s}_j) = q(\mathbf{s}_i)\,q(\mathbf{s}_j)$.

Averaging (4) yields an upper bound on the average probability of error $\bar{P}_e$

$$\bar{P}_e < e^{RT} \int \left[E_{q(s)} p(\mathbf{v}|\mathbf{s}) e^{-\lambda \tilde{m}(\mathbf{v}|\mathbf{s})} E_{q(s)} e^{\lambda \tilde{m}(\mathbf{v}|\mathbf{s})}\right] d\mathbf{v}\,, \tag{6}$$

where

$$E_{q(s)} f(\cdot) = \int q(\mathbf{s}) f(\cdot)\, ds\,, \quad \text{and } R = \frac{\ln M}{T}\,. \tag{7}$$

The next stage in our model simplification is to restrict treatment to the "memoryless" channel. This assumption implies that each component in the vectors $\mathbf{s}_i$ are picked independently with identical probability distribution. This gives rise to the product from

$$p(\mathbf{v}|\mathbf{s}_i) = \prod_{n=1}^{N} p(v_n|s_{ni})\,, \tag{8}$$

where the v_n's and s_{ni}'s are the nth components of the vectors $\mathbf{v}$ and $\mathbf{s}_i$ respectively and $p(v_n|s_{ni})$ is the nth component probability density. This is a Discrete Memoryless Channel (DMC) model without feedback. As a consequence of these arguments the following also holds,

$$E_{q(s)} P(\mathbf{v}|\mathbf{s}_i) = \prod_{n=1}^{N} E_{q(s_{ni})} p(v_n|s_{ni}) = \left[E_{q(s)} p(v_n|s)\right]^N \tag{9}$$

where, $p(v_n|s)$ and $q(s)$ are now one-dimensional probability densities in case of baseband transmission and two-dimensional, or complex valued, when passband transmission (QAM or L-ary PSK) is used. We also make the reasonable assumption that the metrics $\tilde{m}(\mathbf{v}|\mathbf{s}_i)$, $i = 1, \ldots, M$ are additive, i.e.,

$$\tilde{m}(\mathbf{v}|\mathbf{s}_i) = \sum_{n=1}^{N} m(v_n|s_{ni})\,. \tag{10}$$

The integer values assumed by the metrics $m(v_n|s_{ni})$ are now taken from the finite set of integers $1, 2, \ldots, Q$. In this "memoryless" notation, the signal length becomes,

$$T = N\tau \tag{11}$$

where τ is the duration of the basic signaling interval.

Substituting (8)–(11) into (6) we obtain an upper bound in the form,

$$\bar{P}_e \ge e^{-N\tau(\tilde{R}(\lambda)-R)}\,, \tag{12}$$

where

$$\tilde{R}(\lambda) = -\frac{1}{\tau} \ln \int dv \left[\mathbf{E}_{q(s)} p(v|s)\, e^{-\lambda m(v|s)}\, \mathbf{E}_{q(s)}\, e^{\lambda m(v|s)}\right]. \tag{13}$$

Since λ is an arbitrary positive number we may optimize it to yield a tighter upper bound. Thus,

$$\bar{P}_e \le e^{-N\tau(\tilde{R}(\lambda^*)-R)}\,, \quad \text{and } \tilde{R}(\lambda^*) = \max_{\lambda>0} \tilde{R}(\lambda)\,. \tag{14}$$

It is easy to verify that the objective function $\tilde{R}(\lambda^*)$ is invariant to scaling and shifting of the metric set. To qualify as an objective function, we require that for any set of metrics, there is an assignment $m(v|s)$, where s is a random variable with a prior probability density $q(s)$, such that $\tilde{R}(\lambda^*)$ is greater than zero. Note that when $\lambda = 0$

$$\tilde{R}(0) = -\frac{1}{\tau} \ln \int dv \left[\mathbf{E}_{q(s)}\, p(v|s)\, e^{-\lambda m(v|s)}\right]\Bigg|_{\lambda=0} = 0\,. \tag{15}$$

Clearly, $\left.\frac{d\tilde{R}(\lambda)}{d\lambda}\right|_{\lambda=0}$ must be positive for at least one metric assignment in order to insure the existence of a nonzero rate. This requirement places the following constraint on the allowed set of metric assignments.

$$\left.\frac{d\tilde{R}(\lambda)}{d\lambda}\right|_{\lambda=0} = \int dv \Big[E_{q(s)}\, m(v|s)\, p(v|s) - E_{q(s)} m(v|s)\, E_{q(s)}\, p(v|s)\Big] > 0\,. \tag{16}$$

Without loss of generality[2] we can assume that 0 and 1 are in the metric set, and that the metric assignment is according to the equation

$$m(v|s_i) = \begin{cases} 1 & \text{if } p(v|s_i)\, p(s_i) > p(v|s_j)\, p(s_j)\ \forall \quad j \neq i \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

A necessary and sufficient condition for $\left.\frac{d\tilde{R}(\lambda)}{d\lambda}\right|_{\lambda=0} = 0$, is that for every value of v,

$$m(v|s_i) = 1 \rightarrow p(v|s_i) = \sum_{j=1}^{L} q(s_j)\, p(v|s_j) = p(v)\,,\ \forall\ v \in \mathbf{V} \tag{18}$$

[2] By scaling and shifting any metric one can generate an equivalent metric set with 0 and 1.

which is equivalent to a system with zero mutual information $\left(I(\mathbf{v} : \mathbf{s}) = 0\right)$. Thus, for any memoryless communication system with $I(\mathbf{v} : \mathbf{s}) > 0$ there exist a non-zero rate.

Now, suppose that in our formulation, we allow the metrics $m(v|s)$ to be unbounded. In this case $m(v|s)$ can very closely approximate $p(v|s)$. Moreover, we see from Schwartz's inequality,

$$\mathbf{E}_{q(s)}\, p(v|s)\, e^{-\lambda m(v|s)}\, \mathbf{E}_{q(s)}\, e^{\lambda m(v|s)} \geq \left[\mathbf{E}_{q(s)} \sqrt{p(v|s)}\right]^2 \tag{19}$$

and therefore,

$$\int dv \left[\mathbf{E}_{q(s)}\, p(v|s)\, e^{-\lambda m(v|s)}\, \mathbf{E}_{q(s)}\, e^{\lambda m(v|s)}\right] \geq \int dv \left[\mathbf{E}_{q(s)} \sqrt{p(v|s)}\right]^2 . \tag{20}$$

The right-hand-side of (20) leads directly to the definition of the computational cut-off rate, R_0, [1].

$$R_0 = \max_{q(s)} \left[-\frac{1}{\tau} \ln \int dv \left[\mathbf{E}_{q(s)} \sqrt{p(v|s)}\right]^2\right] . \tag{21}$$

It is straightforward to see from (13) that this rate is achieved when the metrics

$$m(v|s) = \frac{1}{\lambda} \ln \sqrt{p(v|s)} . \tag{22}$$

In this limiting case, we can at best achieve R_0 and clearly, this is not the largest achievable rate, since by using more sophisticated bounding techniques [5] it is possible to achieve channel capacity.

Returning to (13) we observe that for any set of mismatched metrics $\{m(v|s)\}$ the generalized cut of rate is defined as [1]

$$\tilde{R}(\lambda^*, q^*) = \max_{\lambda>0,\, q>0} \tilde{R}(\lambda, q) \tag{23}$$

where $\int q(s)\, ds = 1$.

Thus it may appear reasonable to further maximize (23) with respect to q. This augmented constraint optimization problem is unfortunately intractable and additional simplifications must be made. Invoking the fact that in most practical systems the signal points in a constellation are drawn from a discrete ensemble and so, a reasonable assumption to make is, that if there are L points in the constellation, the probability of drawing such a point in a random code is $q(s) = 1/L$. When these assumptions are applied in (13) we obtain the simplified and workable objective function.

$$\begin{aligned} W\left(\mathbf{m}(v), \lambda\right) &\triangleq e^{-\tau \tilde{R}(\lambda)} \\ &= \int dv \left[\frac{1}{L} \sum_{i=1}^{L} p(v|\mathbf{s}_i)\, e^{-\lambda m(v|s_i)}\right] \left[\frac{1}{L} \sum_{i=1}^{L} e^{\lambda m(v|s_i)}\right] \end{aligned} \tag{24}$$

where $\mathbf{m}(v) = \left(m(v|s_1), m(v|s_2), \ldots, m(v|s_L)\right)$ is the metric L-tuple. Since each $m(v|s_i) \in \{1, 2, \ldots, Q\}$ one of Q^L L-tuples is associated with each point v.

Our central problem can now be stated as follows. Find $\lambda^* > 0$ and metric assignment $\mathbf{m}^*(v)$ for all v such that

$$W\left(\mathbf{m}^*(v), \lambda^*\right) = \min_{m(v),\, \lambda>0} W\left(\mathbf{m}(v) \lambda\right) \tag{25}$$

where each entry in the L-tuple assumes values in the set of integers, $\{1, 2, \ldots, Q\}$.

In the next section, we develop an algorithm which provides a solution to this problem and prove its existence.

III. The Optimization Algorithm

Before deriving the optimum design algorithm and proving the existence of a solution, we examine in some detail feasible and reasonable metric assignments in some limiting situations. We hope that these examples will provide some intuition as to the nature of the optimum assignments in general.

To fix ideas, consider a situation where the channel characterizing probability densities, $p(v|s_i)$, $i = 1, 2, \ldots, L$ are sharply peaked at the signal points s_i. These points are the elementary alphabet from which code words are composed and can be either on the real-line, a plane or in some higher dimensional fixed space. The general ideas illustrated here are independent of the dimensionality of the basic alphabet subspace. The situation we have in mind represents large SNR operations so that the chance of confusing the signal points is very small. In this situation, where the number of points L is fixed, there is hardly a need for coding and the channel capacity approaches $\frac{1}{\tau} \log_2 L$ bits/sec. In this limiting operating situation we expect the cut-off rate as well as the generalized cut-off rate the approach capacity. Moreover, we expect the metric assignments in this idealized situation to be simple and require no more than two integers to specify the metric vector.

To see the rationale for this solution, consider the Voronoi regions of the space of the basic alphabet vectors v. These regions form a partition of the subspace and are defined as follows. Denote the regions formed by the set C_n, $n = 1, 2, \ldots, L$,

$$C_n = \left\{v : v \in \mathbf{R}\ (\text{or } \mathbf{R}^2);\ p(v|s_n) \geq p(v|s_i)\ \forall i\right\} \tag{26}$$

and write (24) as sums of integrals over these regions,

$$\begin{aligned} W\left(\mathbf{m}(v), \lambda\right) = \frac{1}{L^2} \sum_{n=1}^{L} \int_{C_n} dv & \left[\sum_{i=1}^{L} p(v|s_i)\, e^{-\lambda m(v|s_i)}\right] \\ & \left[\sum_{i=1}^{L} e^{\lambda m(v|s_i)}\right] . \end{aligned} \tag{27}$$

In the discussion to follow we assume that the metric set is $\Theta = \{0, 1\}$ and make the following metric assignments.

$$m(v|s_n) = \begin{cases} 1 & v \in C_n \\ 0 & v \notin C_n \end{cases} \quad n = 1, \ldots, L . \tag{28}$$

In other words, the various L-tuples characterizing the metrics are now the following

$$\begin{array}{l} v \in C_1 \rightarrow m(v) = (1,0,0,\ldots,0) \\ v \in C_2 \rightarrow m(v) = (0,1,0,\ldots,0) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ v \in C_L \rightarrow m(v) = (0,0,0,\ldots,1) \end{array}$$

For this assignment we obtain explicitly

$$\begin{aligned} W\Big(\mathbf{m}(v), \lambda\Big) &= \frac{1}{L^2}\left[e^{\lambda}+L-1\right]\sum_{n=1}^{L}\int_{C^n} dv \\ &\quad \left[\sum_{i=1}^{L} p(v|s_i)\, e^{-\lambda m(v|s_i)}\right] \\ &= \frac{1}{L}\left[e^{\lambda}+L-1\right]\left[\left(e^{-\lambda}-1\right)P_c+1\right]. \end{aligned} \tag{29}$$

Note that in this special case the term $\left[\sum_{i=1}^{L} e^{\lambda m(v|s_i)}\right]$ is equal to $\left[e^{\lambda}+L-1\right]$ for all v, and the quantity,

$$P_c = \frac{1}{L}\sum_{n=1}^{L}\int_{C_n} p(v|s_n)\, dv \quad , \tag{30}$$

can be thought of as the uncoded probability of making correct decisions and, as will be seen, is the single parameter determining the value of (29). This can be seen after the derivative of $W\Big(\mathbf{m}(v), \lambda\Big)$ with respect to λ is set to zero, which then results, upon inspection, in a unique minimum for $W\Big(\mathbf{m}(v), \lambda\Big)$ when

$$e^{2\lambda} = \frac{P_c}{1-P_c}(L-1)\,. \tag{31}$$

Substituting, this quantity into (29), we obtain the following optimized figure-of-merit for this particular metric assignment:

$$W\Big(\mathbf{m}(v), \lambda^*\Big) = \frac{1}{L}\left[\sqrt{P_c}+\sqrt{(L-1)/(1-P_c)}\right]^2 \equiv W(P_c) \tag{32}$$

and the generalized cut-off rate for this example becomes,

$$\begin{aligned} \tilde{R}(\lambda^*) &= -\tfrac{1}{\tau}\log_2 W(P_c) = \tfrac{1}{\tau \ln 2}\Big\{\ln L - 2\ln \\ &\quad \left[\sqrt{P_c}+\sqrt{(L-1)(1-P_c)}\right]\Big\} \quad \text{bits/sec.} \end{aligned} \tag{33}$$

We see from (30) that

$$\frac{1}{L} \le P_c \le 1\,. \tag{34}$$

The upper and the lower limit results when $p(v|s_i)$ is independent of s_n. The highest possible rate is achieved when $P_c = 1$ and zero rate results when $P_c = \frac{1}{L}$. These can be verified directly from (33). It can be easily shown that $\tilde{R}(\lambda^*)$ is equal to the computation cut-off rate of a symmetric discrete memory channel with input $s \in \{s_1, s_2, \ldots, s_L\}$ and output $y \in \{y_1, y_2, \ldots, y_L\}$, with transition probability

$$p(y_i|s_j) = \begin{cases} P_c & i = j \\ \frac{1-P_c}{L-1} & i \ne j \end{cases} \quad . \tag{35}$$

In Table I, we have calculated some typical achievable rates for different values of P_c and L. We see that for a binary system with an uncoded error rate of 10^{-2} slightly more than 25% of capacity is given up with this most elementary binary metric assignment. However, when $L = 4$ a smaller amount of the capacity is given up for the same uncoded error rate. This is not surprising, since $\tilde{R}(\lambda^*)$ is a monotonically increasing function of L that is asymptotically approaching the limit

$$\tilde{R}(\lambda^*) \approx -\frac{1}{\tau}\log_2(1-P_c) \quad . \tag{36}$$

Let us denote by E_s the signal energy and by N_o the

TABLE I

SOME TYPICAL ACHIEVABLE RATES FOR DIFFERENT VALUES OF P_c AND L

L	$\frac{E_s}{N_o}\frac{1}{\log(L)}$ [dB] BPSK/QPSK	$1-P_c$	$\tilde{R}\left[\frac{\text{bits}}{\tau \log_2 L}\right]$	$W(P_c)$
2	0.50	6.7 10^{-2}	0.41478	0.75013
2	3.00	2.3 10^{-2}	0.62256	0.64952
2	3.24	2.0 10^{-2}	0.64386	0.64000
2	4.32	1.0 10^{-2}	0.73817	0.59950
2	6.79	1.0 10^{-3}	0.91157	0.53161
2	8.40	1.0 10^{-4}	0.97143	0.51000
4	0.50	1.3 10^{-1}	0.36165	0.60571
4	3.00	4.5 10^{-2}	0.57186	0.45259
4	3.24	4.0 10^{-2}	0.59436	0.43869
4	4.32	2.0 10^{-2}	0.69627	0.38090
4	6.79	2.0 10^{-3}	0.89373	0.28968
4	8.40	2.0 10^{-4}	0.96523	0.26235
4	4.32	2.0 10^{-2}	0.69561	0.38124
4	5.21	1.0 10^{-2}	0.77572	0.34117
4	7.33	1.0 10^{-3}	0.92375	0.27787
4	8.79	1.0 10^{-4}	0.97530	0.25871
4	9.59	2.0 10^{-5}	0.98888	0.25388

single-sided additive noise spectral density. Then, from the observation of Table I it is clear that for a fixed value of $\frac{E_s}{N_o \log(L)}$ (fixed energy per dimension), the objective function decreases while L increases due to the suboptimal metric assignment for $L = 4$.

Now, let us assign the metric $\{0,1\}$in a more efficient way with respect to the objective function (24). From the observation of Eq. (24) it is clear that the right term can only assume L different values. i.e.

$$\sum_{i=1}^{L} e^{\lambda m(v|s_i)} = re^{\lambda}+(L-r),\; r \in \{0,1,\ldots,L-1\}\,. \tag{37}$$

Here we exclude the all ones metric, since it is equivalent to the all zeros metric. We also can conclude that if $p(v|s_i) > p(v|s_j)$ then $m(v|s_i) > m(v|s_j)$, since otherwise we can reduce the objective function by exchanging the metrics. Thus, the received signal space v is partitioned into $2^L - 1$ subsets. Now, let us denote by $B_r(v)$ the set of r most likely signals, i.e.

$$B_r(v) = \left\{ s_i | p(v|s_i) > p(v|s_j) \text{ for at least } L - r \text{ distinct } s_j \right\} \tag{38}$$

For a fixed value of λ, we have to assign the metric vector with r^* ones, that minimizes the function

$$\left[e^{-\lambda} \sum_{i \in B_r(v)} p(v|s_i) + \sum_{i \notin B_r(v)} p(v|s_i) \right] \left[r e^{\lambda^2} + (L - r) \right] . \tag{39}$$

The optimal metric assignment required to find λ^*, that minimize equation (24), where the metric assignment is according to equations (38) and (39).

As it becomes evident, the explicit demonstration of the optimum assignment even in the simplest of situations is not tractable and can only be done by the use of a search algorithm.

We now return to the objective function (24) and wish to determine a $\lambda \geq 0$ and a metric vector $\mathbf{m}(v) = \left(m(v|s_1), m(v|s_2), \ldots, m(v|s_L) \right)$ among all Q^L tuples which minimize it. Since the integrand is positive, the minimization can be carried out for each point v. Thus, our problem reduces to the following integer programming minimization problem. Find, the solution to

$$\min_{\lambda > 0,\, m(v) \in Q^L} \sum_{i,j}^{L} p(v|s_i)\, e^{-\lambda \left[m(v|s_i) - m(v|s_j) \right]} \tag{40}$$

for a given set of metric assignments $\{m^*(v|s_i)\}_1^L$ when

$$\begin{aligned} &\sum_{i,j}^{L} p(v|s_i)\, e^{-\lambda \left[m^*(v|s_i) - m^*(v|s_j) \right]} \\ \leq \; &\sum_{i,j}^{L} p(v|s_i)\, e^{-\lambda \left[m(v|s_i) - m(v|s_j) \right]} , \\ &\text{for all } v \in \mathbf{R} (\text{or } \mathbf{R}^2) . \end{aligned} \tag{41}$$

This induces, for each λ, a complete partition of the v space into at most Q^L disjoint sets, $\{A_n\}_1^{Q^L}$. Once having performed this partition for a given λ, we repeat it for all positive λ. To complete the optimum assignment, we choose the $m(v|s_i)$ and the λ satisfying (41) and yielding the smallest value of (24). The algorithm can be implemented in the signal space or in the likelihood domain. It can be easily shown that under certain conditions the boundary between quantization regions in the likelihood space is an hyperplane. Similar result was obtained by Lee [4] when the quantization is done under different criteria.

Four remarks about the above algorithms are in order. First, the most time consuming part of the algorithm is the selection of metric assignment for each point. This step can run faster, if the search is over a smaller set of candidates. One possible set of candidates is the set of metrics of the adjacent regions to this point in the previous iteration. Second, for the simple case of binary signaling the problem can be simplified as it is done in the first example in section IV. Third, this algorithm can be used either for continuous conditional density functions, or for discrete conditional distributions. This is a major advantage of this algorithm since the numerical computations are done based on discrete distributions. Massey [3], algorithm as well as Lee [4] algorithm cannot be used for discrete conditional distributions. Fourth, for most practical problems (QAM, L-ary PSK, non orthogonal modulation), the dimension of signal space is smaller (typically 2) than the dimension of the likelihood space which is $L - 1$. In addition, an algorithm in the signal space is simpler and does not require the mapping from the likelihood space to the signal space (see Lee's [4] remark regarding the evaluation of the cost function). Based on these remarks we think that the algorithm in the signal space is more efficient than is counterpart in the likelihood space.

While the foregoing description of the design algorithm is straightforward, two fundamental questions must be answered. First, what is the computationally complexity of the search algorithm, and second does a solution exist. We now elaborate on the first question. We envision a decoder that will have a table look-up consisting of a list of the metric vector to be assigned to each received point v. Since v is assumed to be continuous, the table look-up will provide metrics to a finely quantized version of the $\mathbf{v}$ space. Thus, the complexity of the decoder will consist only of adding integer values from the set $(1, 2, \ldots, Q)$ after looking up corresponding values in the table look-up. The design task is to construct a look-up table. As to the second question, we actually answered by the example we have presented earlier. There, we demonstrated that there exist acceptable metric assignments for a unique $\lambda \geq 0$ and in fact have calculated the resulting generalized cut-off rates for such assignment.

In the next section, we provide additional illuminating examples and show how to apply the search procedures outlined above. Also, we work out in detail the $L = 2$ case which can be done in closed form.

IV. Special Cases and Numerical Examples

In this section we will examine a few examples where our approach can be easily used to find a good metric assignment. We will start with the simple binary case for which the problem is reduced to finding a minimum value of a polynomial.

A. The binary case, L=2

Here for illustration purposes the analysis is restricted to a binary channel with a continuous channel output random variable, v, characterized by some symmetric conditional probability density functions $f(v| \pm a)$, $v \in (-\infty, \infty)$. In this case Eq. (24) can be now written as

$$\begin{aligned} W\Big(m(\mathbf{v}), \lambda\Big) &= \int dv \Big[\tfrac{1}{2} \Big(f(v|a)\, e^{-\lambda m(v|a)} + f(v|-a) \\ & e^{-\lambda m(v|-a)} \Big) \Big] \cdot \Big[e^{\lambda m(v|a)} + e^{\lambda m(v|-a)} \Big] \\ &= 1 + \tfrac{1}{2} \int dv \Big[f(v|a)\, e^{-\lambda \Delta(v)} \\ & + f(v|-a)\, e^{\lambda \Delta(v)} \Big] \end{aligned} \tag{42}$$

where the difference metric $\Delta(v) = m(v|a) - m(v|a)$ takes on values in the set $\Theta = \{0, \pm 1, \ldots, \pm k, \ldots, \pm 2Q\}$. Let us now fix λ. Then for any value of v we have to assign a pair of metrics k or $-k$, which corresponds to the transmitted signals a or $-a$, respectively. In order to minimize our objective function, given $\lambda > 0$, we select Δ that minimizes the integrand in Eq. (24). In other words, the metric k will be assigned to the vector v, if k is the solution to the inequality,

$$f(v|a)\zeta^{-k} + f(v|-a)\zeta^{k} \leq f(v|a)\zeta^{-j} + f(v|-a)\zeta^{-j}, \ \forall j \in \Theta, \tag{43}$$

where $\zeta = e^{\lambda}$. This inequality can be expressed as a function of the likelihood ratio

$$\Lambda(v) = \frac{f_{v|a}(v)}{f_{v|-a}(v)}, \text{i.e., } \Lambda(v)(\zeta^{-k} - \zeta^{j}) \leq \zeta^{j} - \zeta^{k}, \ \forall j \in \Theta.$$

For a given value of $\lambda > 0$, the optimal metric assignment is to assign the metric k for all received signal v that satisfies the inequality

$$\zeta^{2k-1} < \Lambda(v) \leq \zeta^{2k+1}. \tag{44}$$

We note that any value of λ induces a unique partition of the received signal space to $(4Q+1)$, nonoverlapping distinct sets on the real line. Now the minimization of the objective function is reduced to find the optimal parameter $\lambda^* = \ln(\zeta^*)$.

A concrete example let us determine the optimal quantization for a BPSK coded system operating over a Gaussian channel. In this case $\Lambda(v) = \exp(2va)$ and $a^2 = \frac{2E_s}{N_o}$, where E_s is the signal energy and N_o is the single-sided additive noise spectral density.

The optimal quantizer, in the sense of minimizing Eq. (24), will assign a metric k or $-k$ to v according to

$$m(v) = k \operatorname{sign}(v), \ |v| \in \Big((2k-1)\delta/2, (2k+1)\delta/2\Big], \ k \geq 0, \tag{45}$$

where $\delta = \ln(\zeta)/a = \lambda/a$, and $\zeta = e^{\lambda}$.

For a coded BPSK system with $E_s/N_o = 0.5$ dB, and the metric set $\Theta = (0, \pm 1, \pm 2, \pm 3, \pm 4)$, the optimal metric assignment over a Gaussian channel is a uniform quantizer with a spacing $2\delta = 0.55$ (or $\lambda = \delta a = .55 \cdot \sqrt{2 \cdot 10^{(E_s/N_o)/10}} = 0.82$). This result agrees with the upper bound on the error performance of a coded system which was obtained by the generating function technique, and the simulation results in Heller and Jacobs [7]. Figure 2 depicts the minimal E_s/N_o required to support τ/R_o (or $\tau/\tilde{R}(\lambda^*)$) for infinity quantization, and two practical metric sets. From these result we conclude that the loss due to metric constraint is a less than .2 dB for most practical rates. Figure 3 describes the region size for these metric sets as a function of E_s/N_o. Note that for the metric set $\Theta = (0, \pm 1, \pm 2, \pm 3, \pm 4)$, the region for zero metric is $[-\delta, \delta]$ and for the metric set $\Theta = (0, \pm 1, \pm 3, \pm 5, \pm 7)$, the region for $+1$ metric is $[0, 2\delta]$.

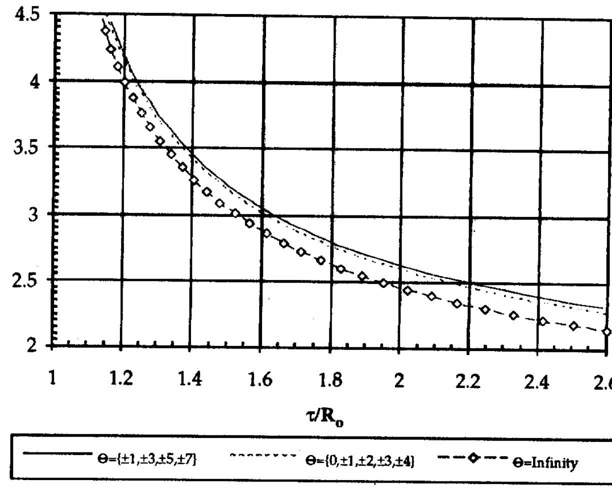

Fig. 2. Minimum E_s/N_o E_b/N_0 as a function of $1/R_o$ for various quantization metrics.

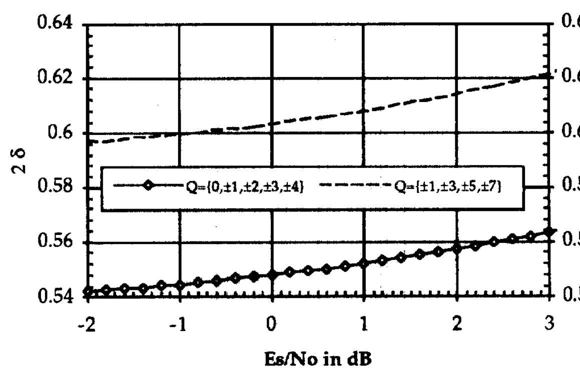

Fig. 3. Threshold value as a function of E_s/N_o and metric set.

B. L-ary PSK modulation over a Gaussian Channel

Let us now assume a Gaussian channel where the received signals are two dimensional vectors $v = (x, y)$, with conditional probability density function.

$$\begin{aligned} p\Big((x,y)/(a,b)\Big) &= \tfrac{1}{2\pi} \exp\left(\tfrac{(x-a)^2}{2} - \tfrac{(y-b)^2}{2} \right), \\ & a^2 + b^2 = 2E_s/N_o, \end{aligned} \tag{46}$$

where (a, b) are the coordinates of the signals in the 2–D signal space. For L-ary PSK modulation the signal constellation is symmetric with respect to rotation of $2\pi/L$ and reflection. Thus, it is sufficient to find the optimal metric assignment for all vectors inside an angel of π/L. Clearly for a fixed value of v, one has to sort the conditional probabilities $p(v|s_i)$ in decreasing order (i.e., sorting the signal points in increasing angular distance), and to assign their metric in decreasing order. For example for $\phi \in [0, \pi/L)$, and signal set $s_i = \sqrt{a^2+b^2}\, e^{2i\pi/L}$, $i = 0, \ldots, L-1$, the metrics satisfy the inequality

$$\begin{aligned} m(v|s_0) \;\geq\; & m(v|s_1) \geq m(v|s_{L-1}) \geq m(v|s_2) \\ & \geq \cdots \geq m(v|s_{L/2}) . \end{aligned} \tag{47}$$

The vector $\mathbf{m}(v)$ assumes $f_L(|\Theta|)$ different values for $\phi \in [0, \pi/L)$ where the function $f_L(|\Theta|)$ is expressed in a recursive form,

$$f_L(|\Theta|) = \sum_{i=1}^{|\Theta|} f_{L-1}(|\Theta| + 1 - i),\; f_2(j) = j,\; j = 1, 2, \ldots, \tag{48}$$

where $|\Theta|$ is the cardinality of the set Θ. The function $f_L(|\Theta|)$ is given in Table II for different values of L. Note that the metric $(0,0,0,0)$ is equivalent to the metric (m, m, m, m).

For QPSK, $E_s/N_o = 3.5$ dB, the partition of the signal space is shown in Figure 4. We observe that our objective function, $\tilde{R}/\tau$, assumes a value $\tilde{R}/\tau = 0.48$ that is larger than was obtained from the simple quantization of the space that was proposed in section III (Table I). Thus we see the gain our metric assignment yields is about 20% more in $\tilde{R}/\tau$ with regard to the standard hard decision for BPSK.

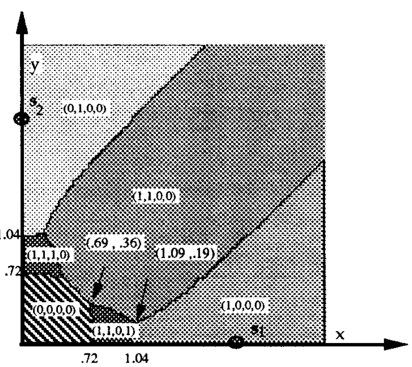

Fig. 4. Partition of the first quadrant for QPSK , Q=2.

V. The Error Performance of a Coded System

For binary signaling and convolutional codes, the error performance can be evaluated via the generating function approach [5]. In this case, the generating function can be obtained as the transfer function of the state diagram of the code regarded as a signal flow graph. The state transitions are then labeled as $J^i W^r$, where i denotes the corresponding number of channel symbols, and r is the Hamming distance between the error path corresponding to the state transition and the correct path corresponding to the all-zeros codeword. Zehavi and Wolf [8] proved that under certain conditions the same state diagram (but with different labels) can be used to compute the error weight distribution of a class of trellis codes. Following the same approach, upper bounds on the error performance of L-ary modulation with quantization can be computed for a special class of codes and channels. The following definitions differ slightly from the original ones in [8] and are useful in the generalization to follow.

Let us assume a trellis coded system which is based on a rate $(n-1)/n$ convolutional encoder followed by a mapping μ that maps the n-tuples of the encoder $\mathbf{C}$ to signals $\in \{s_1, \ldots, s_L\}$. As we stated before the quantizer can be viewed as a partitioning of received signal space into regions. If the received signal falls in a region A_j then the quantizer assigns a vector of metric

$$\mathbf{m}(A_j) \equiv \Big(m_j(s_1),\, m_j(s_2), \ldots, m_j(s_L)\Big) = \mathbf{m}(v),\; \forall v \in A_j \tag{49}$$

As before $\mathbf{m}(A_j)$ assumes values in a finite set of vectors. To evaluate performance of the mismatched metric for a trellis coded system we need the following definitions.

Definition 1: Let $\mathbf{E}$ be binary n-tuple. Then, the profile of the signal $s = \mu(\mathbf{C})$ with respect to a vector $\mathbf{E}$ is

$$F(\mathbf{C}, \mathbf{E}, W) = \sum_{A_j} P_j(\mathbf{C}) W^{m_j(\mu(\mathbf{C} \oplus \mathbf{E})) - m_j(\mu(\mathbf{C}))} \tag{50}$$

where $P_j(\mathbf{C})$ stands for the conditional probability Prob $(v \in A_j | \mu(\mathbf{C})$ was transmitted), and $\oplus$ is the mod 2 addition operation,

Definition 2: Let B be a set of channel signals of cardinality 2^{n-1}. The weight profile of the set B with respect to a given n-tuple vector $\mathbf{E}$ is denoted $F(B, \mathbf{E}, W)$, and is given by

$$F(B, \mathbf{E}, W) = \sum_{\mu(\mathbf{C}) \in B} F(\mathbf{C}, \mathbf{E}, W) . \tag{51}$$

Definition 3: Let us assume that B and $B^c = \Omega - B$, are two distinct subsets composed of channel signals. Then, the combined signal constellation and the channel has uniform weight profile property with respect to subset B if

$$F(B^c, \mathbf{E}, W) = F(B, \mathbf{E}, W) = F(\mathbf{E}, W),\; \forall \mathbf{E} . \tag{52}$$

Let us also assume that B and B^c are distinct subsets composed of channel signals which diverge from the same state in the trellis diagram of the code, and that the combined signal constellation and the channel has uniform weight profile with respect to subset B. Then, for a trellis code that satisfies the conditions of Theorem 1 in [8][3] the modified generating function approach can be used for bounding the error performance of the code, by selecting the value

[3] There are two conditions:
(a) The trellis code is based upon a binar (linear) convolutional code followed by a nonlinear mapping from the encoder output to channel input symbols.
(b) The uniform weight profile property holds (Eq. (52)).

TABLE II
SOME VALUES OF $f_L(Q)$ FOR DIFFERENT VALUES OF L

L	$f_L(\|\Theta\|)$	$\mathbf{m}(v)$, for $Q = 2$
1	1	(0)
2	Q	(00), (10)
3	$\frac{Q(Q+1)}{2}$	(000), (110), (100)
4	$\frac{Q(Q+1)(Q+2)}{6}$	(0000), (1110), (1100), (1000)
8	$\frac{Q(1+Q)(2+Q)}{2520} \cdot (5021280 - 7516422Q + 1704521Q^2 + 1076532Q^3 - 285491Q^4$	(00000000), (11111110), (11111100) , (11111000), (11110000), (11100000), (11000000), (10000000)

of W that minimizes the value of the modified generating function. Note, that uniform weight profile property depends on the code, the signal constellation and the transition probabilities of the channel.

The modified generating function of a trellis code enumerates the number of codewords that have a fixed pattern of weight profile. The modified generating function of a trellis code, $T(W, J)$ can be written as a sum of products of the weight profiles [8] given by

$$T(W, J) = \sum_{\mathbf{E}}^{N} \prod_{p=1}^{N} JF(\mathbf{E}, W) \,, \tag{53}$$

Here the sum is all over codewords $\mathbf{E} = \{\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_p, \ldots, \mathbf{E}_N\}$ of the convolutional code that diverge from the all zero state and emerge after N branches. Based on the results of [8], this sum of products can be computed by labeling a state diagram with S states, where S is the number of decoder states if the trellis code is based on a linear convolutional code and the code has uniform weight profile property.

For a given codeword $\mathbf{E}$ the polynomial $\prod_{p=1}^{N} F(\mathbf{E}_p, W)$ can be written as

$$G(\mathbf{E}, W) = \prod_{p=1}^{N} F(\mathbf{E}_p, W) = \prod_{r=1}^{2^n} F(E = \mathbf{C}_r, W)^{n_r} \,. \tag{54}$$

Here, n_r is the number of times the error sequence $\mathbf{E}$ possess the vector $E = \mathbf{C}_r$.

The sum of all pairwise error probabilities for choosing an incorrect path (with respect to a vector $\mathbf{E}$) is equal to the total sum of the coefficients of $G(\mathbf{E}, W)$ with negative metrics plus half of the coefficient with zero metric.

Let us denote by,

$$A(z) = \sum_{k=-N}^{N} a_k Z^k \text{ and } \{A(z)\}_- = a_o/2 + \sum_{k=-N}^{-1} a_k \,. \tag{55}$$

Therefore,

$$\sum_{\mathbf{C} \text{ in the code}} P_E(\mathbf{C} \Rightarrow \mathbf{C} \oplus \mathbf{E}) = \{G(\mathbf{E}, W)\}_- \tag{56}$$

It is clear that $\{A(z)\}_-$ cannot be easily calculated. Therefore, we can use the Chernoff bound approach. Let $F(E = \mathbf{E}_r, W_r) = \min_{w>0} F(E = \mathbf{E}_r, W)$. Then, in a similar manner to Viterbi and Omura [5, pp. 291–292], the average first error event is bounded by

$$\bar{P}_E \leq \{T(W, J)\}_-\big|_{J=2/L} \leq \frac{1}{2} \min_r \{T(W_r, J)\}\Big|_{J=2/L} \,. \tag{57}$$

Another approach is to use $W^* = e^{\lambda^*}$, where λ^* is the optimal Chernoff parameter that minimizes the objective function in Eq. (24).

Thus, a tighter upper bound on the first error event probability of a trellis code that satisfies our conditions, is given by

$$\bar{P}_E \leq \frac{1}{2} \min \left\{ \min_r \{T(W_r, J)\}\Big|_{J=2/L}, \{T(W^*, J)\}\Big|_{J=2/L} \right\} \tag{58}$$

and a similar expression can be obtained for the bit error rate.

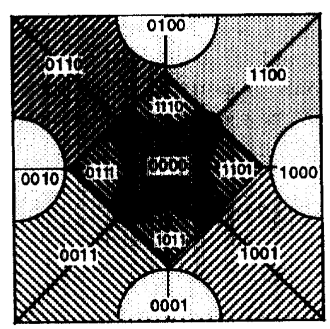

Fig. 5. Schematic description of the signal space partition for QPSK, Q=2.

Consider an example of QPSK signals and $Q = 2$. In this case there are thirteen subsets, A_j, and thirteen different values of the conditional probability, P_j. For a symmetric channel with respect to a phase rotation of 90° shift, there are only nine distinct values of conditional probabilities P_j. These can be easily verified from the schematic diagram in Figure 5 and Table III. The index j of P_j stands for

TABLE III
METRIC ASSIGNMENTS AND CONDITIONAL PROBABILITIES FOR QPSK MODULATION

C	A_0	A_1	A_2	A_3	A_4	A_6	A_7	A_8	A_9	A_{11}	A_{12}	A_{13}	A_{14}
00	0	0	0	0	0	0	0	1	1	1	1	1	1
01	0	0	0	0	1	1	1	0	0	0	1	1	1
11	0	0	1	1	0	1	1	0	0	1	0	0	1
10	0	1	0	1	0	0	1	0	1	1	0	1	0

(a)

C	A_0	A_1	A_2	A_3	A_4	A_6	A_7	A_8	A_9	A_{11}	A_{12}	A_{13}	A_{14}
00	P_0	P_1	P_2	P_3	P_1	P_3	P_7	P_8	P_9	P_{11}	P_9	P_{13}	P_{11}
01	P_0	P_2	P_1	P_3	P_8	P_9	P_{11}	P_1	P_3	P_7	P_9	P_{11}	P_{13}
11	P_0	P_1	P_8	P_9	P_1	P_9	P_{13}	P_2	P_3	P_{11}	P_3	P_7	P_{11}
10	P_0	P_8	P_1	P_9	P_2	P_3	P_{11}	P_1	P_9	P_{13}	P_3	P_{11}	P_7

(b)

TABLE IV
THE WEIGHT PROFILE OF THE SIGNAL $s = \mu(\mathbf{C})$ WITH RESPECT TO $\mathbf{E}$

$C\backslash E$	(00)	(01)	(11)	(10)
(00)	1	$\alpha W^1 + \beta W^{-1} + (1-\alpha-\beta)$	$\gamma W^1 + \rho W^{-1} + (1-\gamma-\rho)$	$\alpha W^1 + \beta W^{-1} + (1-\alpha-\beta)$
(01)	1	$\alpha W^1 + \beta W^{-1} + (1-\alpha-\beta)$	$\gamma W^1 + \rho W^{-1} + (1-\gamma-\rho)$	$\alpha W^1 + \beta W^{-1} + (1-\alpha-\beta)$
(11)	1	$\alpha W^1 + \beta W^{-1} + (1-\alpha-\beta)$	$\gamma W^1 + \rho W^{-1} + (1-\gamma-\rho)$	$\alpha W^1 + \beta W^{-1} + (1-\alpha-\beta)$
(10)	1	$\alpha W^1 + \beta W^{-1} + (1-\alpha-\beta)$	$\gamma W^1 + \rho W^{-1} + (1-\gamma-\rho)$	$\alpha W^1 + \beta W^{-1} + (1-\alpha-\beta)$
$F(E,W)$	2	$2\Big(\alpha W^1 + \beta W^{-1} + (1-\alpha-\beta)\Big)$	$2\Big(\gamma W^1 + \rho W^{-1} + (1-\gamma-\rho)\Big)$	$2\Big(\alpha W^1 + \beta W^{-1} + (1-\alpha-\beta)\Big)$

$\alpha = (P_1 + P_3 + P_7);\ \beta = (P_8 + P_9 + P_{11});\ \gamma = (P_2 + 2P_3);\ \rho = (P_8 + 2P_9).$

the integer number representation of the binary four tuple $\mathbf{m}(v)$, i.e. if A_j has a metric vector $\mathbf{m}(v) = (1\,1\,1\,0)$, then $j = 14$. The weight profile of a signal $\mu(C)$ with respect to any binary vector is not a function of the signal itself, and therefore with respect to the partition of the subsets $B = ((00), (11))$ and $B^c = (01, 10)$, the code has the uniform weight profile property as it is shown in Table IV. Thus, the average first error event probability is bounded by

$$\bar{P}_E \leq \frac{1}{2} \min\Big\{\min_r\{T(W_r, J)\}\Big|_{J=1/2} \cdot \{T(W^*, J)\}\Big|_{J=1/2}\Big\} \tag{59}$$

with

$$W_1 = W_3 = \sqrt{\beta/\alpha};\ W_2 = \sqrt{\rho/\gamma}\ . \tag{60}$$

VI. CONCLUSION

In this paper we have proposed a physically reasonable objective function for selecting the desired assignment of metrics to the received analog signals. We developed a search algorithm for designing a table-look-up that is used by the decoder to select the appropriate intermediate metrics and showed that an optimum solution exists. We provided a number of illuminating examples to elucidate our ideas and have worked out in detail some practical cases. A new bound based on the modified generating approach for quantized coded system was derived and applied for QPSK convolutional coded data transmission.

VII. ACKNOWLEDGMENTS

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